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STATISTICS OF ELECTROMAGNETIC SCATTERING FROM CHAFF CLOUDS

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STATISTICS OF ELECTROMAGNETIC SCATTERING FROM CHAFF CLOUDS

PASSIVE ECM BRANCH
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The problem of electromagnetic scattering from chaff clouds is investigated using statistical methods. This effort is supplemental to other contractual efforts under the sponsorship of the Avionics Laboratory aimed at treating the same problem using deterministic methods.

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20. Abstract

The physical model for a chaff cloud is assumed to be a very large collection of half wavelength linear dipoles spaced sufficiently apart so that mutual interaction or coupling effects can be ignored. Furthermore, signal fluctuations are attributed solely to the phase changes due to dipole movement. Although the effects of homogeneous and isotropic turbulence have been considered in a cursory fashion, no thorough investigations of such factors as wind shear and turbulence in general has been possible.

Starting from first principles, the first and second order probability densities of the scattered field from chaff clouds are derived. Auto-correlation functions and power spectra of the received voltage, radar cross section and phase are obtained. All the mathematical derivations are explained in full detail. For the simple case of a spherically uniform distribution of relative speed of the dipoles, it is shown that an integral relation exists between the speed distribution function and the intensity auto-correlation function. The utility of second order statistics in studying the effects of chaff clutter fluctuations on advanced radars such as moving target indicator is demonstrated. Finally, numerical results are included both from an actual experiment and calculations based on assumed dipole velocities. Although it was not possible to compare the two unrelated events, there were definite trends of similarity in the data.

FOREWORD

This technical report discusses statistical aspects of electromagnetic scattering by chaff clouds and was performed in-house at the Air Force Avionics Laboratory under Project 7633, "Passive Electronic Countermeasures," Task 13, "Aerospace Vehicle Signature Control/Masking", during the period June 1973 to March 1974. The principal investigator for this work was Dr. V. P. Pyati.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SECTION I
INTRODUCTION

The problem of electromagnetic scattering by chaff clouds can be treated in two distinct fashions, namely, deterministic and nondeterministic or statistical. As might be expected, each has its own merits and drawbacks. Deterministic methods predict the exact outcome in any given situation and are generally quite involved. In the case of chaff clouds, there are two separate parts to the problem, the aerodynamic and the electromagnetic. First one computes the orientations and positions of the individual dipoles in the chaff cloud using a generalized six-degree-of-freedom-equations program and then the scattering behavior due to plane wave excitation determined by means of well-documented standard techniques. In each case, a digital computer is essential; but for the digital computer, such calculations would be unthinkable. Furthermore, the effects of environment such as wind shear and turbulence can be included as excitation parameters in the aerodynamics calculations. Deterministic methods in the present case are time-consuming and very expensive, and in view of the basic limitations of computers such as finite memory, etc., the number of dipoles that can be handled cannot possibly exceed a few hundred. These techniques are being investigated by the Air Force Avionics Laboratory through other contractual efforts and will be reported later.

Statistical methods, on the other hand, are not concerned with any one particular situation; they predict in a probabilistic fashion what might happen under a given set of circumstances. The methods are quite general and apply to a variety of problems occurring in physics and engineering. While deterministic methods are severely limited by the number of dipoles that can be handled in the case of chaff, there is no such restriction with statistical methods; in fact, the larger the number of dipoles, the more accurate the predictions become. This report treats the problem of electromagnetic scattering from chaff clouds in a systematic and quantitative manner using statistical methods. The material presented here has been gathered from different sources and put into self-contained form. Certain mathematical derivations have been simplified considerably.

The plan of this report is as follows. First, basic material on probability and random processes is introduced. Then, first and second probability densities of chaff cloud scattering are derived starting from first principles. Relevant averages and auto-correlation are obtained. The physical significance of each random function is explained with illustrations. A self-consistent mathematical model for chaff cloud scattering is developed. This will be a basis for further analytical studies in such important areas as the effects of chaff echo fluctuations on continuous wave, pulse doppler, and MTI radars. Furthermore, we hope to combine statistical and deterministic methods in a judicious manner so as to be able to predict chaff cloud behavior more accurately.

SECTION II
PROBABILITY THEORY

2.1 DEFINITIONS

A random variable (r.v.) that is a function of time is called a random process. Let us denote the r.v. by $x(t)$ and the value attained at an instant of time t_k by

$$x_k = x\left(t_k\right) \tag{2.1}$$

Because of the random nature, it is meaningless to talk about the value attained at a particular instant of time or the values observed over a period of time. Then how does one handle the problem? The answer is, of course, by using the notion of probability which lends to the precise definitions of certain distributions and averages which can be predicted and observed with some measure of confidence. The first and second order probability density functions (PDF) are defined by

$p_1\left(x;t\right)\,dx$ = probability of finding x between x and $x+dx$ at time t

$p_2\left(x_1,x_2;t_1,t_2\right)\,dx_1dx_2$ = joint probability of finding a pair of values x in the ranges $\left(x_1,x+dx_1\right)$ at time t_1 and $\left(x_2,x_2+dx_2\right)$ at time t_2

These definitions can be extended to still higher order PDF, but they will not be needed in our investigations. For convenience of writing, the differential elements will not be carried along with p_1 and p_2, but should always be understood. In the definition of p_2, it should be understood that both x_1 and x_2 are random variables with the same distribution p_1. These two variables are considered statistically independent or simply independent if

$$p_2\left(x_1,x_2;t_1,t_2\right) = p_1\left(x_1,t_1\right)\ p_1\left(x_2,t_2\right) \tag{2.2}$$

$$= p_2\left(x_1,t_1|x_2,t_2\right) p_1\left(x_2,t_2\right) \qquad (2.3)$$

and p_1 and p_2 fulfill the relations

$$\int p_1\left(x_1;t_1\right) dx_1 = 1 \qquad (2.4)$$

$$\int p_2\left(x_1,x_2;t_1,t_2\right) dx_2 = p_1\left(x_1,t_1\right) \qquad (2.5)$$

$$\int p_2\left(x_2;t_2|x_1;t_1\right) dx_2 = 1 \qquad (2.6)$$

In relation to the independent variable time t, random processes are divided into two categories. These are stationary and nonstationary processes according as to whether the statistics are independent of or dependent upon t. The latter processes are extremely complex and will not be considered here. Stationary processes are further subdivided into strictly stationary and wide-sense stationary. For our purposes, it suffices to consider only wide-sense stationary processes. What this means is the following. The first order PDF is independent of time and the second order PDF depends only on the difference $t_2 - t_1 = \tau$. Hence

Such, of course, is not always the case. To handle the dependent case, the notion of conditional probability is introduced. This is denoted by $p_2(x_2,t_2|x_1,t_1)$ which gives the probability of finding x_2 in the range $(x_2,x_2 + dx_2)$ at time t_2 given that $x = x_1$ at time t_1. A vertical bar separates the two sets of variables and the variables appearing on the right side have already occurred and are considered no longer random. One has, by definition

$$\begin{aligned} p_2\left(x_1,x_2;t_1,t_2\right) &= p_2\left(x_2,t_2|x_1,t_1\right) p_1\left(x_1,t_1\right) \\ &= p_2\left(x_1,t_1|x_2,t_2\right) p_1\left(x_2,t_2\right) \end{aligned} \tag{2.3}$$

and p_1 and p_2 fulfill the relations

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$$\int p_2\left(x_1,x_2;t_1,t_2\right) dx_2 = p_1\left(x_1,t_1\right) \tag{2.5}$$

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$$p_1\left(x;t\right) = p_1\left(x\right) \tag{2.7}$$

$$p_2\left(x_1,x_2;t_1,t_2\right) = p_2\left(x_1,x_2;\tau\right) \tag{2.8}$$

for wide-sense stationary processes.

The notion of homegeneity in time is sometimes employed to describe the foregoing random process. One should, of course, not lose sight of the fact that x is still a function of time. From now on, subscripts denoting the order of the PDF will also be omitted for convenience of writing.

The cumulative probability denoted by P(x) which gives the probability that $-\infty < x < a$ is defined by

$$P(a) = \mathrm{Prob}(-\infty<x<a) = \int_{-\infty}^{a} p(x)dx \tag{2.9}$$

from which it follows that

$$p(x) = \frac{d}{dx}P(x) \tag{2.10}$$

The complementary function

$$P_c(x) = 1 - P(x) \tag{2.11}$$

is also commonly used.

The expectation or expected value of a function f(x) is defined as

$$E(f) = \int f(x)p(x)dx \tag{2.12}$$

with the most important ones being

$$E(x) = \int xp(x)dx \tag{2.13}$$

$$E\left(x^2\right) = \int x^2p(x)dx \tag{2.14}$$

The variance and standard deviation (SD) of x become

$$\left.\begin{aligned} Var &= D(x) = E\left(x^2\right) - E^2(x) \\ SD &= \sigma(x) = \sqrt{D(x)} \end{aligned}\right\} \qquad (2.15)$$

If x represents voltage for instance, $E(x)$ is the D.C. component, $E(x^2)$ the mean square and $D(x)$ the A.C. component. In relation to the second order PDF the most significant quantity of interest is the auto-correlation function $B(\tau)$ defined as

$$B(t) = E\left(x_1x_2\right) = E(x(t)x(t+\tau))$$

$$= \iint x_1x_2\ p\left(x_1,x_2;t\right)\ dx_1dx_2 \qquad (2.16)$$

It may be noted that

$$B(o) = E\left(x^2\right) \qquad (2.17)$$

and for large time lags, x_1 and x_2 will be uncorrelated so that

$$B(\infty) = E^2(x) \qquad (2.18)$$

which is again the D.C. component.

The auto-covariance $K(\tau)$ and the normalized version $R(t)$ are defined by

$$K(t) = B(t)-B(\infty) \qquad (2.19)$$

$$R(t) = \frac{K(t)}{K(0)} \qquad (2.20)$$

It may be noted that $-1 \leq R(\tau) \leq 1$. Since the correlation function gives the correlation between $x(t)$ and $x(t + t)$, the more rapidly $x(t)$ changes with time, the more rapidly $R(t)$ decreases from its maximum value of unity. This decrease may be characterized by a correlation time t_o defined by

$$R\left(t_o\right) = 1/e \qquad (2.21)$$

where e is the base of natural logarithim. The foregoing gives us a clue that R(t) and the frequency spectrum must somehow be connected. This indeed is the case and given by the Wiener-Khintchine theorem.

$$\Phi\ (\omega) = \int_0^a R(\tau)\ \cos\ \omega t\ dt \tag{2.22}$$

$$R(t) = \frac{2}{\pi} \int_0^\infty \Phi\ (\omega)\cos\ t\omega\ d\omega \tag{2.23}$$

It is noted that the frequency spectrum Φ and R form a Fourier transform pair. It will not always be possible to integrate (2.22), and in such cases, it is customary to take $\omega_{max} = 2\pi/t_o$.

Now let us introduce the concept of time averages. Suppose there are great numbers of identical radio receivers (ensemble) turned on simultaneously. Let us also assume that the transients have died down and steady conditions have been reached. The noise output voltages of all the receivers are recorded over a long period of time T. At a definite time t_1, we take the voltages $x^{(1)}(t_1)$, $x^{(2)}(t_1),\ldots,$ compute the average and the probability density function. This average is called statistical or ensemble average and this is what we have considered thus far. Stationarity in this context means the statistics are the same regardless of the value of t. We might just as well take the output of a single receiver and define a time average in the customary manner (denoted by a over bar)

$$\overline{x(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt \tag{2.24}$$

From an experimental veiwpoint, it is much easier and more convenient to measure time averages. Naturally we would like to know the relation between time average $\overline{x}(t)$ and ensemble average E(x). Under the so-called ergodic hypothesis, these two are equal. This identity will be invoked here as a basis for comparing theory and experiment. The time average auto-correlation is defined by

$$\overline{B}(t) = \overline{x(t)x(t+\tau)}$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\ x\ (t+\tau)dt \qquad (2.25)$$

taking a single realization of the random process.

Under the ergodic hypothesis for correlation functions

$$B(\tau) = \overline{B}(\tau) \qquad (2.26)$$

2.2 FUNCTIONS OF RANDOM VARIABLES

Suppose

$$y = f(x) \qquad (2.27)$$

and we want to determine the PDF of y. If the inverse function

$$x = f^{-1}(y) \qquad (2.28)$$

is single valued then

$$p_y(y) = p_x\left[f^{-1}(y)\right] \left|\frac{dx}{dy}\right| \qquad (2.29)$$

where subscripts are used to distinguish different functions. In case of multiple values, we first define single valued branches

$$x_1 = f_1^{-1}(y), x_2 = f_2^{-1}(y), \ldots \qquad (2.30)$$

and get the more general formula

$$p_y(y) = p_x\left(x_1\right) \left|\frac{dx_1}{dy}\right| + p_x\left(x_2\right) \left|\frac{dx_2}{dy}\right| \qquad (2.31)$$

$$+ \ldots$$

These ideas are easily extended to functions of several random variables. For example, if x and y are random variables and

$$u = u(x,y), \quad v = v(x,y) \tag{2.32}$$

then in terms of the joint density of x and y

$$p_{uv}(u,v) = p_{xy}[x(u,v),y(u,v)]\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \tag{2.33}$$

where we have assumed the inverse functions

$$x = x(u,v) \quad y = y(u,v) \tag{2.34}$$

are single valued. For multiple values, we proceed as in the case of a single variable and obtain a result similar to (2.31). The last member of Equation 2.33 is called the Jacobian of the transformation

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{2.35}$$

which is part and parcel of the transformation. We must caution that failure to include the Jacobian would lead to erroneous conclusions regarding PDF of the new variables.

2.3 RANDOM SIGNAL WITH UNIFORM PHASE

In order to provide a better feel for the material or probability theory introduced in this chapter, let us consider, for example, the ensemble defined by the sinusoids

$$\underset{\sim}{x}(t,\phi) = A\cos(\omega t+\phi) \tag{2.36}$$

where A, ω are fixed and ϕ distributed uniformly over a complete period. This means

$$p(\phi) = \frac{1}{2\pi}, \quad -\pi\leq\phi\leq\pi \tag{2.37}$$

In practice, $x(t,\phi)$ may typically represent a scattered field, with A, ω, and ϕ representing the amplitude, carrier frequency, and phase, respectively. The various probabilities and expected values can be

determined in the following manner. First we invert Equation 2.36 to obtain the two branches,

$$\phi_1 = \cos^{-1}\left(\frac{x}{A}\right) -\omega t, \quad \phi \le \phi_1 \le \pi$$

$$\phi_2 = \cos^{-1}\left(\frac{x}{A}\right) -\omega t+\pi, \quad \pi \le \phi_2 \le 2\pi$$

Differentiating,

$$\left|\frac{d\phi_1}{dx}\right| = \left|\frac{d\phi_2}{dx}\right| = \frac{1}{\left(A^2-x^2\right)^{1/2}}$$

so that

$$p(x) = \frac{1}{\pi\left(A^2-x^2\right)^{1/2}} \tag{2.38}$$

Note that the result does not depend on ω or t (stationary). As $x \to \pm A$, p(x) becomes infinite, which seems to conflict with the fundamental fact that probability can never exceed unity. The answer lies in realizing that p(x) by itself has no physical meaning unless it is multiplied by the differential element dx, which together give the probability of finding x in the range (x,x + dx). This will never exceed unity. We conclude, therefore, that the signal level is most likely to be found near +A or -A. If A were not a constant, one has

$$p(x) = \frac{1}{\pi} \int \frac{p(A)dA}{\left(A^2-x^2\right)^{1/2}} \tag{2.39}$$

where p(A) is the PDF of A.

For the second order PDF, all that is needed is the conditional probability occurring in Equation 2.3. Since Equation 2.36 is a deterministic function, once its value is known at t_1, it is specified (functionally) for all other times. Thus

$$x_2 = A \cos \left(\omega t_2 + \phi\right)$$

$$= A \cos \left[\omega\tau + \cos^{-1}\left(\frac{x_1}{A}\right)\right] \tag{2.40}$$

with $\tau = t_2 - t_1$, as usual.

Employing the delta function notation to denote the PDF of a constant, we have

$$p\left(x_2 \middle| x_1;\tau\right) = \delta\left[x_2 - A \cos\left(\omega\tau + \cos^{-1}\left(\frac{x_1}{A}\right)\right)\right] \tag{2.41}$$

It is much easier to compute ensemble and time averages, for example,

$$E(x) = \int_{-\pi}^{\pi} x p(\phi) d\phi = 0 \tag{2.42}$$

$$\overline{x} = \overline{A \cos (\omega t + \phi)} = 0 \tag{2.43}$$

$$\overline{x_1 x_2} = \frac{A^2}{2} \cos \omega t \tag{2.44}$$

$$E\left(x_1 x_2\right) = \frac{A^2}{2} \cos \omega t \tag{2.45}$$

We note in particular, the process is not only stationary, but ergodic in the sense

$$E(x) = \overline{x} \tag{2.46}$$

$$E\left(x_1 x_2\right) = \overline{x_1 x_2} \tag{2.47}$$

If the restriction that A be a constant is removed, the second relation will not be true in general, which means that the process fails to be ergodic with respect to the auto-correlation function. Also if Φ is not uniformly distributed the process is no longer stationary. In what follows, we will assume a uniform distribution for ϕ.

SECTION III
SCATTERING FROM CHAFF CLOUDS

3.1 GENERAL CONSIDERATIONS

The word chaff denotes a confusion type electronic countermeasure employing a large number of resonant dipoles. The dipoles are usually in the form of very narrow aluminum strips or aluminum-coated glass fibers cut to a length of about one-half wavelength at the frequency of interest. Since the bandwidth of narrow dipoles is quite small, one generally uses several cuts to obtain coverage over a wide band of frequencies. When properly distributed in space, a chaff cloud may occupy a large volume. At microwave frequencies a single chaff package contains literally hundreds of thousand dipoles.

For a distance R from the transmitter that is large compared to the pulse width T, the number of scatterers or dipoles per resolution cell is

$$N = n \left(\frac{\pi R \theta}{2}\right)^2 \frac{cT}{2} \tag{3.1}$$

where n is the average number of scatterers per unit volume, θ the radar beamwidth, and c the velocity of light. At any instant of time, one may assume that the number of scatterers entering the range cell is equal to that leaving so that N can be considered more or less constant and not a random quantity. Making N random will complicate the problem unnecessarily without altering the conclusions in any substantive manner.

It is obvious that a very large number of elementary targets are involved in the scattering process. The signal scattered by each elementary target will have random phase and amplitude because: (1) the orientation of the dipole may change due to rotation, and (2) the distance between radar and the dipole center may change. The phase of the returned signal is actually independent of the orientation, but the amplitude is not. However, if the rotation rates are small compared to time of observation, the amplitude

changes may be neglected. The dependence of return power on initial orientations can be included by suitably defined PDF for the angles. The second cause can affect amplitude as well as phase. The amplitude changes are quite small and may be ignored. The phase changes are most important and have been studied quite thoroughly. One generally assumes that the phase distributions are uniform over a full cycle. What this means is that a single dipole may occupy any position within the range cell with equiprobability. Also, the number of dipoles wich any given phase will be the same as those with any other phase. The dipoles will also be assumed to be independently moving and the effects of mutual coupling will be neglected. The effects of wind will be examined to some extent. Since we are dealing with noncoherent scattering, mass motion of the cloud with constant speed will have negligible effects because all the individual dipoles are affected equally. However, if there is relative motion between the dipoles either due to turbulence conditions or some other reason, the return signal will fluctuate proportionately. These fluctuation rates are attributable to the doppler beats of the individual scatterers. Using probabilistic methods we will now develop the first and second order statistics. The material has been gathered from several sources shown under references. Some of the derivations, especially second order statistics, are obtained by simpler means.

3.2 FIRST ORDER STATISTICS

For a collection of N scatterers, the resultant complex signal S is given by the vector sum of the individual returns and if one neglects multiple scattering,

$$S = Ve^{i\theta} = \sum_{k=1}^{N} A^{(k)} e^{i\phi^{(k)}} \qquad (3.2)$$

where $A^{(k)}$ is the amplitude and $\phi^{(k)}$ the phase of the k^{th} scatterer and for convenience the additional phase term due to the carrier frequency has been factored out. Our problem is then to find the probabilities of V and θ given the probabilities of $A^{(k)}$ and $\phi^{(k)}$. The above sum represents the familiar random walk problem in the complex plane.

Resolving S into real and imaginary parts, we have

$$\mathrm{Re}\; S = V \cos\theta = x = \sum_{k=1}^{W} A^{(k)} \cos\phi^{(k)} \tag{3.3a}$$

$$\mathrm{Im}\; S = V \sin\theta = y = \sum_{k=1}^{N} A^{(k)} \sin\phi^{(k)} \tag{3.3b}$$

Knowing the joint PDF of $A^{(k)}$ and $\phi^{(k)}$, one might compute the probabilities of the individual terms and the sums by the methods outlined in Chapter II. However, in view of a very powerful theorem called the Central Limit Theorem, there is no need to go about this the hard way.

Central Limit Theorem:

Let $X^{(1)}, X^{(2)}, \ldots$ be N independent random variables all of which have the same distribution with expectation μ and variance σ^2. The distribution of the sum

$$S = \sum_{k=1}^{N} X^{(k)} \tag{3.4}$$

approximates normal for large N with $E(S) = N\mu$ and $D(S) = N\sigma^2$. In other words

$$p(S) = \frac{1}{\sqrt{2\pi N}\,\sigma}\, e^{-\frac{(S-N\mu)^2}{2\sigma^2 N}} \tag{3.5}$$

The beauty of the theorem is that one need not know or be concerned with the individual distributions. It is not known precisely how large N should be, but the conditions are almost always satisfied for chaff clouds. Let us for a moment assume that all the dipoles are cut to the same length (later we shall account for multiple length) and $A^{(k)}$, $\phi^{(k)}$ are uncorrelated which is justified in view of earlier discussions, as a matter of fact they are statistically independent. Let the distribution for phases be

$$p(\phi) = \frac{1}{2\pi}\ , \quad -\pi \leq \phi \leq \pi \tag{3.6}$$

Now

$$\left.\begin{aligned} &E\left[A^{(i)}A^{(j)}\cos\phi^{(i)}\cos\phi^{(j)}\right] = \frac{E\left(A^{(i)^2}\right)}{2}\delta_i^j \\ &\delta_i^j \text{ is Kronecker delta} \end{aligned}\right\} \tag{3.7}$$

and

$$E\left[A^{(k)}\cos\phi^{(k)}\right] = E\left[A^{(k)}\right]E\left[\cos\phi^{(k)}\right]$$

$$= 0$$

Invoking the central limit theorem

$$p(X) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{X^2}{2\sigma^2}} \tag{3.8}$$

where, since the amplitudes have identical PDF, $\sigma^2 = \frac{N}{2}E(A^2)$ and a similar expression for Y, of course. To proceed further, we need the joint PDF of X and Y. In general, it is not possible to write down the joint density from a knowledge of marginal densities. An exception to this is the normal process where all order densities are normal. For the case of two variables, with zero mean and identical variances σ^2, the result is particularly simple,

$$p(X,Y) = \frac{1}{2\pi\sigma^2\left(1-\rho^2\right)^{1/2}}\exp -\left[\frac{X^2-2\rho XY+Y^2}{2\sigma^2\left(1-\rho^2\right)}\right]$$

where the correlation coefficient is

$$\rho = \frac{E(XY)}{\left[E\left(X^2\right)E\left(Y^2\right)\right]^{1/2}} \tag{3.9}$$

From Equation 3.3, we note that E(XY) = 0, so that

$$p(X,Y) = \frac{e^{-\frac{X^2+Y^2}{2\sigma^2}}}{2\pi\sigma^2}$$

$$= p(X)p(Y) \qquad (3.10)$$

Introducing polar coordinates

$$V = \left(X^2+Y^2\right)^{1/2}, \quad \theta = \tan^{-1}\left(\frac{X}{Y}\right)$$

we have, using Equation 2.33

$$p(V,\theta) = \frac{V}{\pi I_o} e^{-\frac{V^2}{I_o}} \qquad (3.11)$$

and

$$I_o = 2\sigma^2 = NE\left(A^2\right)$$

The marginal densities are obtained by integration as usual.

$$p(V) = \frac{2V}{I_o} e^{-\frac{V^2}{I_o}} \qquad (3.12)$$

$$o \leq V < \infty$$

$$p(\theta) = \frac{1}{2\pi}, \quad -\pi \leq \theta \leq \pi \qquad (3.13)$$

The resultant phase is uniform and the distribution for V is called Rayleigh or χ with two degrees of freedom. These results have been known for about 80 years and apply to many kinds of scattering situations. The amplitude distribution of the elementary signals does not

enter into the picture and it is therefore quite erroneous to assume that the dipoles in a cloud should have spherically uniform orientations to arrive at the above statistics. The requirement of uniform phase is essential, however. For nonuniform phase distributions, the analysis is very similar, but the results become quite complicated. The important expectations in the present case are

$$\left.\begin{aligned} E(V) &= \frac{\sqrt{\pi I_o}}{2} \\ E\left(V^2\right) &= I_o \end{aligned}\right\} \tag{3.14}$$

$$\left.\begin{aligned} E(\theta) &= 0 \\ E\left(\theta^2\right) &= \pi^2/3 \end{aligned}\right\} \tag{3.15}$$

The cumulative distribution of V is

$$P(V) = 1 - e^{-V^2/I_o} \tag{3.16}$$

From a practical viewpoint the radar cross section (RCS) of the cloud, defined by $I = V^2$ is more significant. One has

$$p(I) = \frac{e^{-I/I_o}}{I_o}\ , \quad I \geq 0 \tag{3.17}$$

and

$$P(I) = 1 - e^{-I/I_o} \tag{3.18}$$

The distribution is called Rayleigh power or χ^2 with two degrees of freedom. It is easily checked that

$$\left.\begin{aligned} E(I) &= I_o \\ E\left(I^2\right) &= 2I_o^2 \\ SD &= I_o \end{aligned}\right\} \tag{3.19}$$

One normally expresses the SD as a percentage of the expectation which in the present case is 100%. A plot of the distribution of V and I are shown in Figures 1 and 2. (Observe that the most probable value of RCS is zero.) The median is 0.7 I_o and the signal intensity will be one half the average level 39% of the time.

The results obtained thus far are quite general and let us now specialize to chaff clouds. All that remains to be done is to relate I_o to the dipole scattering properties. For a half wave dipole inclined at angles θ and ϕ in a spherical coordinate system, the back scattered amplitude A for a plane wave traveling in the z direction can be approximated by

$$A \approx \sqrt{\sigma_o} \sin^2 \theta \cos^2 \phi \tag{3.20}$$

where in terms of wavelength λ, the broadside RCS $\sigma_o = 0.89\lambda^2$.

What we now need is the joint PDF $p(\theta,\phi)$ describing the orientations of the dipoles. It depends upon the rotation rates, environment, etc., in a complex fashion and no serious attempt has ever been made in this direction. One therefore makes the simplifying a priori assumption that all orientations are equally likely so that

$$p(\theta,\phi) = \frac{\sin \theta}{4\pi}, \quad \begin{aligned} 0 &\leq \phi \leq 2\pi \\ 0 &\leq \theta \leq \pi \end{aligned} \tag{3.21}$$

whence

$$I_o = NE\left(A^2\right) = \frac{N\sigma_o}{5} \tag{3.22}$$

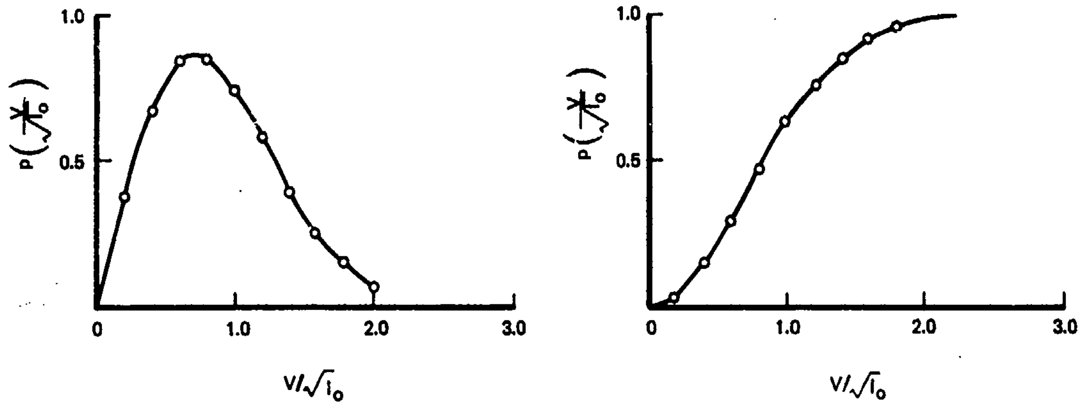

Figure 1. First Probability Distributions of Received Voltage for Random Dipole Clouds

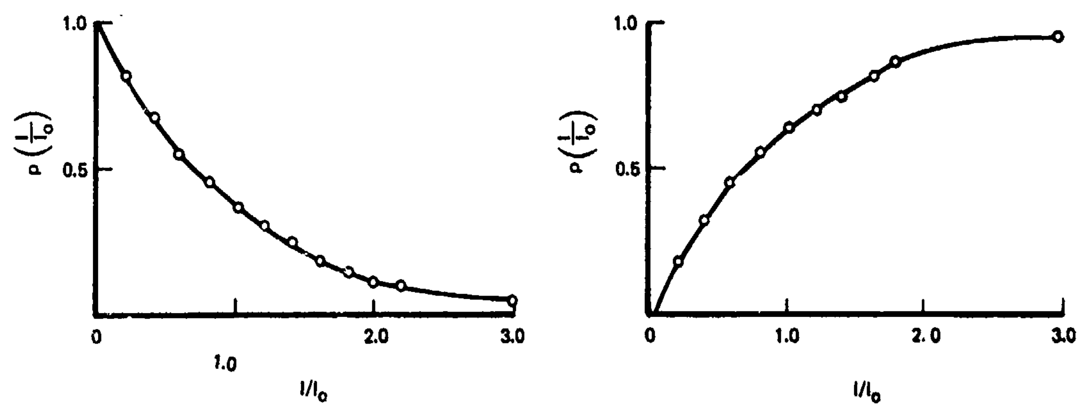

Figure 2. First Probability Distributions of RCS for Random Dipole Clouds

This is the so-called tumble average RCS of the chaff cloud. In case of orthogonal polarization reception, the above expression should be divided by three. For chaff clouds that contain several cuts of dipoles, we simply take

$$I_o = N_1\sigma_1 + N_2\sigma_2 + \dots \tag{3.23}$$

where the N's refer to number and σ's suitably defined averages.

3.2.1 Steady Target Immersed In Chaff

The determination of the radar return of an aircraft flying through a chaff corridor is of considerable practical importance. The radar cross section of an aircraft changes with aspect considerably and there are no reliable statistics on the subject. We will therefore consider the aircraft as a steady target and determine the combined statistics of aircraft plus chaff. This will be at least first step toward understanding a more difficult problem. If the RCS of the steady target is $m^2 I_o$, we have for the probability densities

$$p(V,\theta) = \frac{Ve^{-\left(m^2 + \frac{V^2}{I_o}\right)}}{\pi I_o} e^{\frac{-2Vm}{\sqrt{I_o}} \cos\theta} \tag{3.24}$$

Integrating over θ

$$p(V) = \frac{2Ve^{-\left(m^2 + \frac{V^2}{I_o}\right)}}{I_o} J_o\left(i\,\frac{2Vm}{\sqrt{I_o}}\right) \tag{3.25}$$

and also

$$p(I) = e^{-\left(m^2 + \frac{I}{I_o}\right)} J_o\left(i2m\sqrt{\frac{I}{I_o}}\right) \tag{3.26}$$

where J_o is zeroth order Bessel function. The distribution for phase is quite complicated and will not be given here. The above is called the Rice-Nakagami distribution originally discovered in the study of fading of radio signals due to mixed propagation paths. Using the known integrals,

$$\frac{e^{\beta/\alpha}}{\alpha} = \int_o^\infty e^{-\alpha x} J_o \left(i2\sqrt{\beta x}\right) dx \qquad (3.27)$$

$$\frac{e^{\beta/\alpha}}{\alpha^5}\left[2\alpha^2 + 4\alpha\beta + \beta^2\right] = \int_o^\infty X^2 e^{-\alpha x} J_o\left(i2\sqrt{\beta x}\right) dx \qquad (3.28)$$

we get

$$E(I) = \left(1+m^2\right)\left(I_o\right) \qquad (3.29)$$

which could have been guessed and

$$E\left(I^2\right) = I_o^2\left(2 + 4m^2 + m^4\right) \qquad (3.30)$$

$$SD = I_o\left(1 + 2m^2\right)^{1/2} \qquad (3.31)$$

$$\frac{SD}{E(I)} = \frac{\left(1 + 2m^2\right)^{1/2}}{1 + m^2} \qquad (3.32)$$

$$\simeq \sqrt{2}/m, \text{ for } m^2 >> 1$$

Summarizing we note that RCS statistics of a chaff cloud with uniform distribution of dipoles follow X^2 distribution with two degrees of freedom. If there is a nonfluctuating target in addition, we have a Rice-Nakagami distribution. The fluctuations are quite large (100%) in the first case and the presence of a strong steady target decreases the fluctuations.

3.3 SECOND ORDER STATISTICS

Whereas the first order statistics tell us about the magnitude of fluctuations, information regarding the rates of fluctuations is obtained from the second order statistics which give the probability of jointly finding two values of a random variable at different times. Analogous to Equation 3.3, we define the joint density

$$p\left(X_1, X_2; t_1, Y_1, Y_2; t_2\right) \tag{3.33}$$

where $X_1 = X(t_1)$, $X_2 = X_2(t_2)$, etc. First, we assume the process is stationary so that time appears as only the difference $\tau = t_2 - t_1$. Also since X and Y are statistically independent, we have

$$p\left(X_1, X_2; Y_1, Y_2; \tau\right) = p\left(X_1, X_2; \tau\right) p\left(Y_1, Y_2; \tau\right) \tag{3.34}$$

We first observe that the marginal densities of X_1 and X_2 are normal and thus given by Equation 3.8. Therefore, analogous to Equation 3.9 we have

$$p\left(X_1, X_2; \tau\right) = \frac{\exp - \left[\dfrac{X_1^2 - 2gX_1X_2 + X_2^2}{I_o\left(1-g^2\right)}\right]}{\pi I_o\left(1-g^2\right)^{1/2}} \tag{3.35}$$

where instead of the correlation coefficient, we have the correlation function of the process given by

$$g(\tau) = \frac{E\left(X_1 X_2\right)}{\left[E\left(X_1^2\right) E\left(X_2^2\right)\right]^{1/2}} \tag{3.36}$$

By symmetry, identical expressions are valid for the Y-component. More will be said about g(t) later, but now a few observations are in order. At $\tau = 0$, the two variables are fully correlated and at the other extreme when $\tau = \infty$, the variables are independent. In mathematical terms

$$\left.\begin{aligned} p\left(X_1, X_2; 0\right) &= p\left(X_1\right) \delta\left(X_2 - X_1\right) \\ &= p\left(X_2\right) \delta\left(X_1 - X_2\right) \end{aligned}\right\} \tag{3.37}$$

$$p\left(X_1, X_2; \infty\right) = p\left(X_1\right) p\left(X_2\right) \tag{3.38}$$

Determination of the various densities and expectations now becomes a routine matter although the algebra becomes very tedious at times.

First we introduce polar-coordinates

$$V_i = \left(X_i^2 + Y_i^2\right)^{1/2}, \quad \theta_i = \tan^{-1}\left(\frac{Y_i}{X_i}\right), \quad i = 1,2$$

to obtain

$$p\left(V_1,V_2,\theta_1,\theta_2;\tau\right) = \frac{V_1V_2}{\pi^2 I_o^2\left(1-g^2\right)}$$

$$\exp -\left[\frac{V_1^2+V_2^2-2gV_1V_2\cos\left(\theta_1-\theta_2\right)}{I_o\left(1-g^2\right)}\right] \tag{3.39}$$

$$o\leq V_1,V_2<\infty, \quad -\pi\leq\theta_1,\theta_2\leq\pi$$

Since the above cannot be written in the product form

$$p\left(V_1,V_2\right) p\left(\theta_1,\theta_2\right)$$

the amplitude V and phase θ are correlated.

Integrating over the angles

$$p\left(V_1,V_2;\tau\right) = \frac{4V_1V_2}{I_o^2\left(1-g^2\right)} J_o\left[i\,\frac{2gV_1V_2}{I_o\left(1-g^2\right)}\right]$$

$$\exp -\left[\frac{V_1^2+V_2^2}{I_o\left(1-g^2\right)}\right] \tag{3.40}$$

If we denote the ratio $V_2|V_1$ by Q

$$p\left(Q,V_2;\tau\right) = \frac{4QV_2^2}{I_o^2\left(1-g^2\right)} J_o\left[i \frac{2gQV_2^2}{I_o\left(1-g^2\right)} \right] \exp - \left[\frac{V_2^2\left(1+Q^2\right)}{I_o\left(1-g^2\right)} \right] \tag{3.41}$$

and

$$p(Q;\tau) = \int_o^\infty p\left(Q,V_2;\tau\right) dV_2 \tag{3.42}$$

using the integral

$$\int_o^\infty Xe^{-\alpha X} J_o(i\beta X)dx = \frac{\alpha}{\left(\alpha^2-\beta^2\right)^{3/2}} \tag{3.43}$$

the distribution is found to be

$$p(Q;\tau) = \frac{2Q\left(1-g^2\right)\left(1+Q^2\right)}{\left[\left(1+Q^2\right)^2-4g^2Q^2\right]^{3/2}} \qquad 0\leq Q<\infty \tag{3.44}$$

and is plotted in Figure 3. Note that as $g \to 1$, it peaks at the center like a delta function. Similarly one has for the intensities

$$p\left(I_1, I_2; \tau\right) = \frac{\exp - \left[\dfrac{I_1 + I_2}{I_o\left(1-g^2\right)}\right]}{I_o\left(1-g^2\right)} J_o\left[i \frac{2g\sqrt{I_1, I_2}}{I_o\left(1-g^2\right)}\right]$$

$$0 \leq I_1, I_2 < \infty \qquad (3.45)$$

and for the ratio I_2/I_1, denoted by W,

$$p(W;\tau) = \frac{\left(1-g^2\right)(1+W)}{\left[(1+W)^2 - 4g^2 W\right]^{3/2}} \qquad (3.46)$$

$$0 \leq W < \infty$$

This distribution is shown in Figure 4. To get the joint density for θ, we integrate Equation 3.39 over V_1 and V_2. First we introduce polar-coordinates

$$V_1 = \rho \cos \psi, \; V_2 = \rho \sin \psi$$

so that

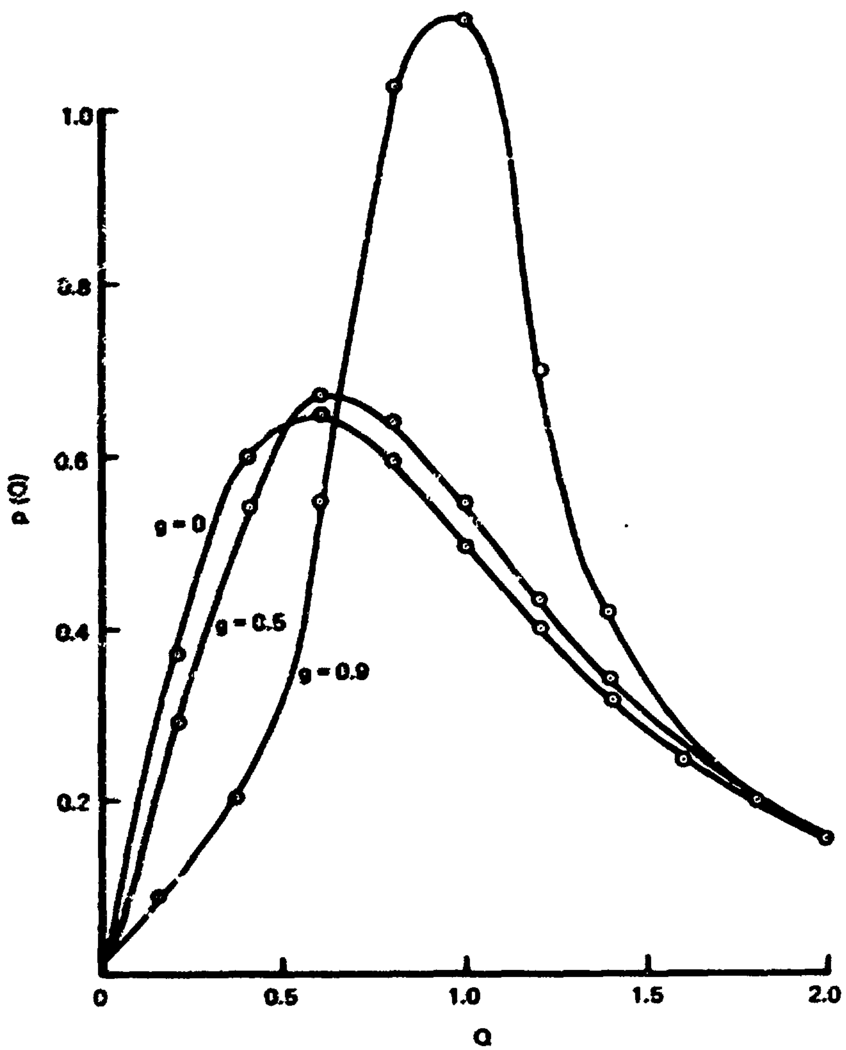

Figure 3. PDF of Voltage Ratio Q

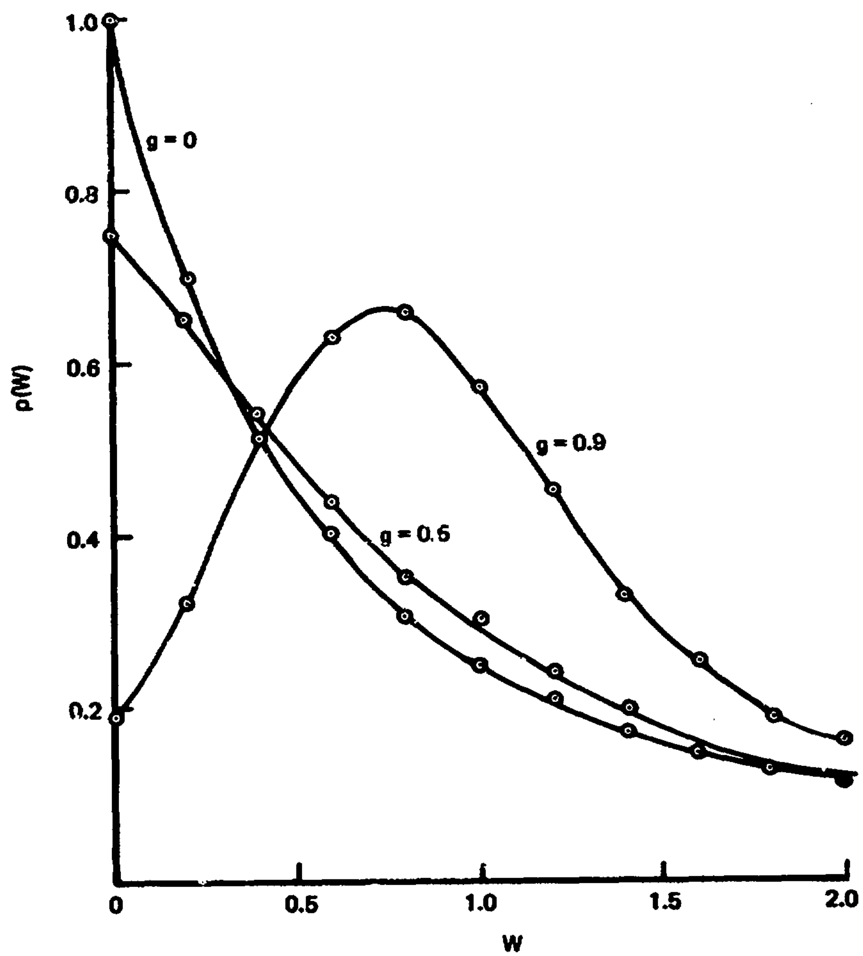

Figure 4. PDF of RCS Ratio W

$$p\left(\theta_1,\theta_2;\tau\right) = \frac{1}{2\pi^2 I_o^2\left(1-g^2\right)} \int_o^{\pi/2} \sin 2\psi \, d\psi$$

$$\int_o^\infty \rho^3 \exp - \left[\frac{\rho^2(1-\gamma \sin 2\psi)}{I_o\left(1-g^2\right)} \right] d\rho$$

with $\gamma = g \cos\left(\theta_1 - \theta_2\right)$

Noting that

$$\int_o^\infty X^3 e^{-\alpha^2 X^2} \, dx = \frac{1}{2\alpha^4}$$

the above becomes

$$\frac{1-g^2}{4\pi^2} \int_o^{\pi/2} \frac{\sin 2\psi}{(1-\gamma \sin 2\psi)^2} \, d\psi$$

$$= \frac{1-g^2}{8\pi^2} \int^\pi \frac{\sin X}{(1-\gamma \sin X)^2} \, dX$$

$$= \frac{1-g^2}{8\pi^2} \frac{d}{d\gamma} \int_o^\pi \frac{dX}{(1-\gamma \sin X)}$$

and using the result

$$\int_0^{\pi} \frac{dX}{(1-\gamma \sin X)} = \frac{\pi + 2 \sin^{-1}\gamma}{\left(1-\gamma^2\right)^{1/2}}$$

we finally have

$$p\left(\theta_1,\theta_2;\tau\right) = \frac{1-g^2}{4\pi^2}\left[\frac{1}{1-\gamma^2} + \frac{\gamma}{\left(1-\gamma^2\right)^{3/2}}\left(\frac{\pi}{2} + \sin^{-1}\gamma\right)\right] \tag{3.47}$$

where

$$\gamma = g(\tau) \cos\left(\theta_1-\theta_2\right)$$

$$-\pi \leq \theta_1, \theta_2 \leq \pi$$

A more convenient expression is obtained by introducing a new set of variables

$u = \theta_1-\theta_2$, $v = \theta_2$, with the limits

$-\pi \leq u+v \leq \pi$ and $-\pi \leq v \leq \pi$

The total probability for u is

$$p(u;\tau) = \int_{\alpha}^{\beta} p(u,v;\tau)dv$$

where care must be exercised in choosing the limits of integration. Manipulating the inequalities for u and v, we note

$$\alpha = \mathrm{Max}(-\pi,-\pi-u)$$
$$\beta = \mathrm{Min}(\pi,\pi-u)$$

which leads to two possibilities

$$u>o,\ \alpha = -\pi,\ \beta = \pi-u$$
$$u<o,\ \alpha = -\pi-u,\ \beta = \pi$$

Hence

$$p(u;\tau) = \frac{1-g^2}{4\pi^2}(2\pi - |u|)\left[\frac{1}{\left(1-g^2 \cos^2 u\right)} + \frac{g \cos u}{\left(1-g^2 \cos^2 u\right)^{3/2}}\left\{\frac{\pi}{2} + \sin^{-1}(g \cos u)\right\}\right]$$

$$-2\pi \le u \le 2\pi \qquad (3.48)$$

A very useful alternate form due to Middleton[4] is

$$p(u;\tau) = \frac{2\pi - |u|}{4\pi^2} \left[1 + g \frac{d}{du} F(u) \right]$$

where

$$F(u) = \frac{\left[\frac{\pi}{2} + \sin^{-1}(g \cos u) \right] \sin u}{\left(1 - g^2 \cos^2 u\right)^{1/2}}$$

The PDF of u is plotted in Figure 5. This completes the derivation of first and second order statistics for the amplitude, intensity (RCS) and phase of the scattered signal by a chaff cloud.

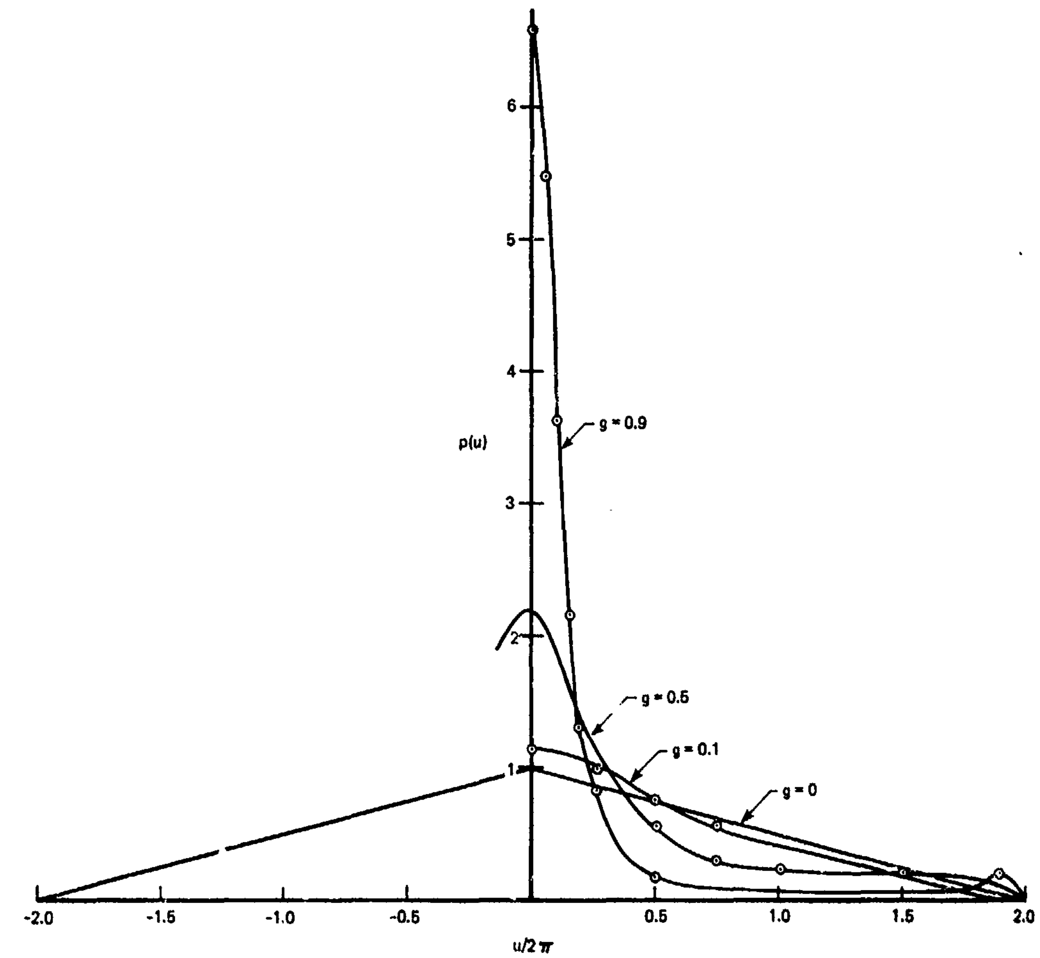

Figure 5. PDF of Phase Difference u

SECTION IV
CORRELATION FUNCTIONS

4.1 CLOUD CORRELATION FUNCTION $g(\tau)$

The correlation function $g(\tau)$ defined by Equation 3.36 is about the most significant quantity characterizing the second order statistics. It is governed by the dynamics of the chaff cloud itself and only in special cases is it possible to establish the connection between the movement of chaff dipoles and $g(\tau)$. Now if $A_1^{(k)}$, $\phi_1^{(k)}$ and $A_2^{(k)}$, $\phi_2^{(k)}$ denote the amplitude and phase of the k^{th} scatterer at times t_1 and $t_2 = t_1 + \tau$, then one has for the total field

$$X_1 = \sum_{k=1}^{N} A_1^{(k)} \cos \phi_1^{(k)} \tag{4.1}$$

$$X_2 = \sum_{k=1}^{N} A_2^{(k)} \cos \phi_2^{(k)} \tag{4.2}$$

Simple calculations show that

$$E\left(X_1^2\right) = \frac{N}{2} E\left(A_1^2\right) \tag{4.3}$$

$$E\left(X_2^2\right) = \frac{N}{2} E\left(A_2^2\right) \tag{4.4}$$

assuming identical scatterers. Furthermore

$$E\left(X_1 X_2\right) = \sum_{i=1}^{N} \sum_{j=1}^{N} E\left(A_1^{(i)} A_2^{(j)}\right) E\left(\cos \phi_1^{(i)} \cos \phi_2^{(j)}\right) \tag{4.5}$$

Since the scatterers are independent

$$E\left(\cos \phi_1^{(i)} \cos \phi_2^{(j)}\right) = 0, \; i \neq j$$

contributions occurs only for i = j. Therefore

$$g(\tau) = \frac{2E\left(A_1A_2\right) E\left(\cos\ \phi_1\ \cos\ \phi_2\right)}{\left[E\left(A_1^2\right) E\left(A_2^2\right)\right]^{1/2}} = 2E\left(\cos\ \phi_1\ \cos\ \phi_2\right) \tag{4.6}$$

if we assume $A_1 = A_2$ which is in keeping with the understanding that amplitude changes are not significant. In terms of the change in phase between the two instants of time, let

$$\psi = \phi_2 - \phi_1, \quad -\infty<\psi<\infty \tag{4.7}$$

so that

$$\begin{aligned} g(\tau) &= 2E\left[\cos\ \phi_1\ \cos\left(\psi+\phi_1\right)\right] \\ &= E(\cos\ \psi) \end{aligned} \tag{4.8}$$

An identical result will be obtained by considering the Y-component. If a dipole moved a distance ξ in the direction of the radar during the time interval τ, then

$$\psi = \frac{4\pi}{\lambda}\ \xi \tag{4.9}$$

where λ is the wavelength. If all directions of motion are equally likely arising from, say, an isotropic turbulent wind field, then the joint density is

$$\begin{gathered} p(V,\theta,\phi) = \frac{1}{4\pi}\ \sin\ \theta q(V) \\ 0\le V<\infty,\ 0\le\theta\le\pi,\ 0\le\phi\le 2\pi \end{gathered} \tag{4.10}$$

where V is the speed and θ, ϕ suitably defined angles with $\theta = 0$ denoting direction away from the radar. Also note that

$$\int_0^\infty q(V)dV = 1 \tag{4.11}$$

Since $\xi = V\tau \cos \theta$, we have

$$\begin{aligned} p(\xi,V,\theta,\phi) &= p(\xi|V,\theta,\phi)\ p(V,\theta,\phi) \\ &= \delta(\xi - V\tau \cos \theta) p(V,\theta,\phi) \end{aligned} \tag{4.12}$$

and therefore

$$p(\xi)d\xi = \frac{d\xi}{2} \int_0^\pi \sin \theta d\theta \int_0^\infty q(V)\delta(\xi - V\tau \cos \theta)dV \tag{4.13}$$

Following Siegert,[5] we express the delta function as a Fourier integral

$$\delta(\xi - V\tau \cos \theta) = \frac{1}{2\pi} \int_{-0}^\infty d\mu e^{i\mu(\xi - V\tau \cos \theta)} \tag{4.14}$$

so that, after integrating over θ,

$$p(\xi) = \frac{1}{2\pi} \int_{-\infty}^\infty d\mu e^{i\mu\xi} \int_0^\infty dV q(V) \frac{\sin(\mu V\tau)}{\mu V\tau} \tag{4.15}$$

and using Equation 4.9

$$p(\psi) = \frac{\lambda}{8\pi^2} \int_{-\infty}^{\infty} d\mu e^{\frac{i\mu\lambda\psi}{4\pi}} \int_{0}^{\infty} dVq(V) \frac{\sin(\mu V\tau)}{\mu V\tau} \tag{4.16}$$

and

$$g(\tau) = E(\cos\psi) = \int_{-\infty}^{\infty} \cos\psi\, p(\psi)d\psi \tag{4.17}$$

The integrations over μ and ψ can be carried out in the following manner.

$$\frac{\lambda}{8\pi^2} \iint_{-\infty}^{\infty} d\mu d\psi \cos\psi\, e^{\frac{i\mu\lambda\psi}{4\pi}} \frac{\sin(\mu V\tau)}{(\mu V\tau)}$$

$$= \frac{\lambda}{2\pi^2} \iint_{0}^{\infty} \cos\psi \cos\left(\frac{\mu\lambda\psi}{4\pi}\right) \frac{\sin(\mu V\tau)}{(\mu V\tau)} d\psi d\mu$$

$$= \frac{\lambda}{4\pi^2} \int_{0}^{\infty} d\mu \frac{\sin(\mu V\tau)}{\mu V\tau} \int_{0}^{\infty} d\psi \left[\cos\left(1 + \frac{\mu\lambda}{4\pi}\right)\psi + \cos\left(1 - \frac{\mu\lambda}{4\pi}\right)\psi\right]$$

$$= \frac{\lambda}{4\pi} \int_{0}^{\infty} d\mu \frac{\sin(\mu V\tau)}{\mu V\tau} \left[\delta\left(1 + \frac{\mu\lambda}{4\pi}\right) + \delta\left(1 - \frac{\mu\lambda}{4\pi}\right)\right] \tag{4.18}$$

$$= \frac{\sin(4\pi V\tau/\lambda)}{(4\pi V\tau/\lambda)}$$

and finally

$$g(\tau) = \int_0^\infty q(V)\,\frac{\sin(4\pi V\tau/\lambda)}{(4\mu V\tau/\lambda)} \qquad (4.19)$$

We may now make the following observations. If a chaff cloud is measured simultaneously at different frequencies and $g(\tau)$ is plotted against $\tau/_{\lambda}$, all the curves should coincide. From a frequency domain viewpoint, it means the doppler beats are proportional to the carrier frequency which is a well known fundamental result. By inverting Equation 4.19

$$q(V) = \frac{2V}{\pi}\left(\frac{4\pi}{\lambda}\right)^2 \int_0^\infty \tau g(\tau)\sin(4\pi V\tau/\lambda)d\tau \qquad (4.20)$$

which may be utilized to estimate the speed distribution from the measured value of $g(\tau)$.

4.2 SIGNAL CORRELATION FUNCTIONS

Using the definition (Equation 2.16), we have for the intensity (RCS)

$$B(I,\tau) = \iint_0^\infty I_1 I_2\, p(I_1,I_2;\tau)dI_1 dI_2$$

$$= I_o^2\left(1+g^2\right) \qquad (4.21)$$

and the auto-covariance

$$R(I,\tau) = g^2(\tau) \qquad (4.22)$$

This is an interesting result because it provides an indirect method of determination $g(\tau)$ from the RCS auto-correlation function. The latter can be determined experimentally using a noncoherent pulse radar. For the amplitude we have

$$B(V,\tau) = \iint_0^\infty V_1 V_2 \, p(V_1, V_2; \tau) dV_1 dV_2$$

using Equation 3.40, and introducing new variables

$$V_1 = t^{1/2} \cos(\phi/2), \; V_2 = t^{1/2} \sin(\phi/2)$$

we have

$$B(V,\tau) = \frac{1}{4I_o^2\left(1-g^2\right)} \int_o^\pi d\phi \sin^2 \phi$$

$$\int_o^\infty dt \, t^2 J_o \left[\frac{igt \sin \phi}{I_o\left(1-g^2\right)}\right] \exp - \left[\frac{t}{I_o\left(1-g^2\right)}\right]$$

$$= \frac{I_o\left(1-g^2\right)^2}{4} \int_o^\pi \sin^2 \phi \left[\frac{3}{\Delta^5} - \frac{1}{\Delta^3}\right] d\phi$$

by use of Equation 3.43, and letting $\Delta = (1 - g^2 \sin^2 \phi)^{1/2}$.

Using elliptic integrals

$$B(V,\tau) = \frac{I_o}{2} \left[2E(g) - \left(1-g^2\right) K(g)\right] \qquad (4.23)$$

where K and $\tilde{E}$ are complete elliptic integrals of the first and second kind with modulus g. Furthermore, the auto-covariance is

$$R(V,\tau) = \frac{2\tilde{E}(g) - \left(1-g^2\right) K(g)}{2-\pi/2} \qquad (4.24)$$

To determine $B(\theta,\tau)$ consider the second moment of $u = (\theta_1-\theta_2)$

$$E\left(u^2\right) = E\left[\left(\theta_1-\theta_2\right)^2\right]$$

$$= 2E\left(\theta^2\right) - 2E\left(\theta_1\theta_2\right)$$

$$= \frac{2\pi^2}{3} - 2B(\theta,\tau) \qquad (4.25)$$

from which

$$B(\theta,\tau) = \frac{\pi^2}{3} - \frac{E\left(u^2\right)}{2} \qquad (4.26)$$

Now from Equation 3.49

$$E\left(u^2\right) = \int_{-2\pi}^{2\pi} u^2 p(u;\tau)\,du$$

$$= \frac{1}{\pi}\int_0^{2\pi} dX\left(X^2 - \frac{X^3}{2\pi}\right)\left[1+g\frac{d}{dX}F(X)\right] \qquad (4.27)$$

with

$$F(X) = \frac{\left[\frac{\pi}{2} + \sin^{-1}(g \cos X)\right] \sin X}{\left(1-g^2 \cos^2 X\right)^{1/2}}$$

The first term gives

$$\int_0^{2\pi} \left(X^2 - \frac{X^3}{2\pi}\right) dX = \frac{2}{3}\pi^3 \tag{4.28}$$

For the second term we integrate by parts repeatedly and find that

$$g \int_0^{2\pi} \left(X^2 - \frac{X^3}{2\pi}\right) \frac{d}{dX} F(X) dX$$

$$= -\pi^2 \sin^{-1} g - \pi\left(\sin^{-1} g\right)^2$$

$$- \frac{1}{2} \int_0^{2\pi} \left(2 - \frac{3X}{2\pi}\right) \left\{ \pi \sin^{-1}(g \cos X) \right.$$

$$\left. + \left[\sin^{-1}(g \cos X)\right]^2 \right\} dX \tag{4.29}$$

The final integrations can be carried out by expanding the inverse sine functions into power series and integrating term by term. First we note below the known results

$$\sin^{-1}(g \cos X) = \sum_{k=0}^{\infty} \frac{(2k)!(g \cos X)^{2k+1}}{2^{2k}(k!)^2(2k+1)}$$

$$\left[\sin^{-1}(g \cos X)\right]^2 = \sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2(g \cos X)^{2k+2}}{(2k+1)!(k+1)}$$

$$\int_0^{2\pi} (\cos X)^{2k+1} dX = 0 = \int_0^{2\pi} X(\cos X)^{2k+1} dX$$

$$\int_0^{2\pi} (\cos X)^{2k+2} dX = \frac{2\pi(2k+2)!}{2^{2k+2}\{(k+1)!\}^2}$$

$$\int_0^{2\pi} X(\cos X)^{2k+2} dX = \frac{2\pi^2(2k+2)!}{2^{2k+2}\{(k+1)!\}^2}$$

Then, it follows that

$$\int_{o}^{2\pi} \sin^{-1}(g \cos X)dX = 0$$

$$\int_{o}^{2\pi} X \sin^{-1}(g \cos X)dX = 0$$

$$\int_{o}^{2\pi} \sin^{-1}(g \cos X)\ {}^2dX = \pi \sum_{n=1}^{\infty} \frac{g^{2n}}{n^2}$$

$$\int_{o}^{2\pi} X\ \sin^{-1}(g \cos X)\ {}^2dX = \pi^2 \sum_{n=1}^{\infty} \frac{g^{2n}}{n^2}$$

Substituting the above result in Equation 4.29 we have finally

$$E\left(u^2\right) = \frac{2\pi^2}{3} - \pi \sin^{-1} g - \left(\sin^{-1} g\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{g^{2n}}{n^2} \tag{4.30}$$

and

$$E\left(\theta_1\theta_2\right) = \frac{1}{2}\left[\pi \sin^{-1} g + \left(\sin^{-1} g\right)^2 - \frac{1}{2} \sum_{n=1}^{\infty} \frac{g^{2n}}{n^2}\right] \tag{4.31}$$

The normalized phase auto-covariance becomes

$$R(\theta,\tau) = \frac{3}{2}\left[\frac{\sin^{-1} g}{\pi} + \left(\frac{\sin^{-1} g}{\pi}\right)^2 - \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{g^{2n}}{n^2}\right] \tag{4.32}$$

4.3 NUMERICAL RESULTS

The problem of determining the speed distribution of the dipoles in a chaff cloud under a given set of environmental conditions is extremely complex and so far no serious attempts have ever been made in this direction. We will therefore be content by presenting results for an assumed distribution. For example, if all the dipoles have the same speed V_o,

$$q(V) = \delta\left(V - V_o\right) \tag{4.33}$$

and from Equation 4.19

$$g(\tau) = \frac{\sin\left(4\pi V_o \tau/\lambda\right)}{4\pi V_o \tau/\lambda} \tag{4.34}$$

The auto-covariance function for the intensity becomes

$$R(I,\tau) = \left[\frac{\sin\left(4\pi V_o \tau/\lambda\right)}{4\pi V_o \tau/\lambda}\right]^2 \tag{4.35}$$

and the frequency spectrum is

$$\Phi(I,\omega) = \frac{\pi}{2\omega_o}\left(1 - \frac{|\omega|}{2\omega_o}\right) \quad |\omega| \le 2\omega_o \tag{4.36}$$

where

$$\omega_o = \frac{4\pi V_o}{\lambda}$$

The spectrum thus contains frequencies up to $4\ V_o/\lambda$ which is no more than the doppler beat between dipoles moving directly into and away from the radar beam. It should be emphasized that we are talking of frequency spread due to, for example, turbulence and not the conventional doppler frequency due to average motion of the entire cloud. The latter cannot be measured by a noncoherent system because the phase information is lost. For the special value of $g(\tau)$ given by Equation 4.34, the three auto-covariance functions are plotted in Figure 6 and Figure 7 shows typical experimental results obtained in a recent AFAL contractual effort.[6] Although the data in the two figures are unrelated, one notices certain trends. For the type of chaff payloads employed, the correlation times were found to be in the order of 10-20 milliseconds which means that the frequency components are around 50-100 Hz.

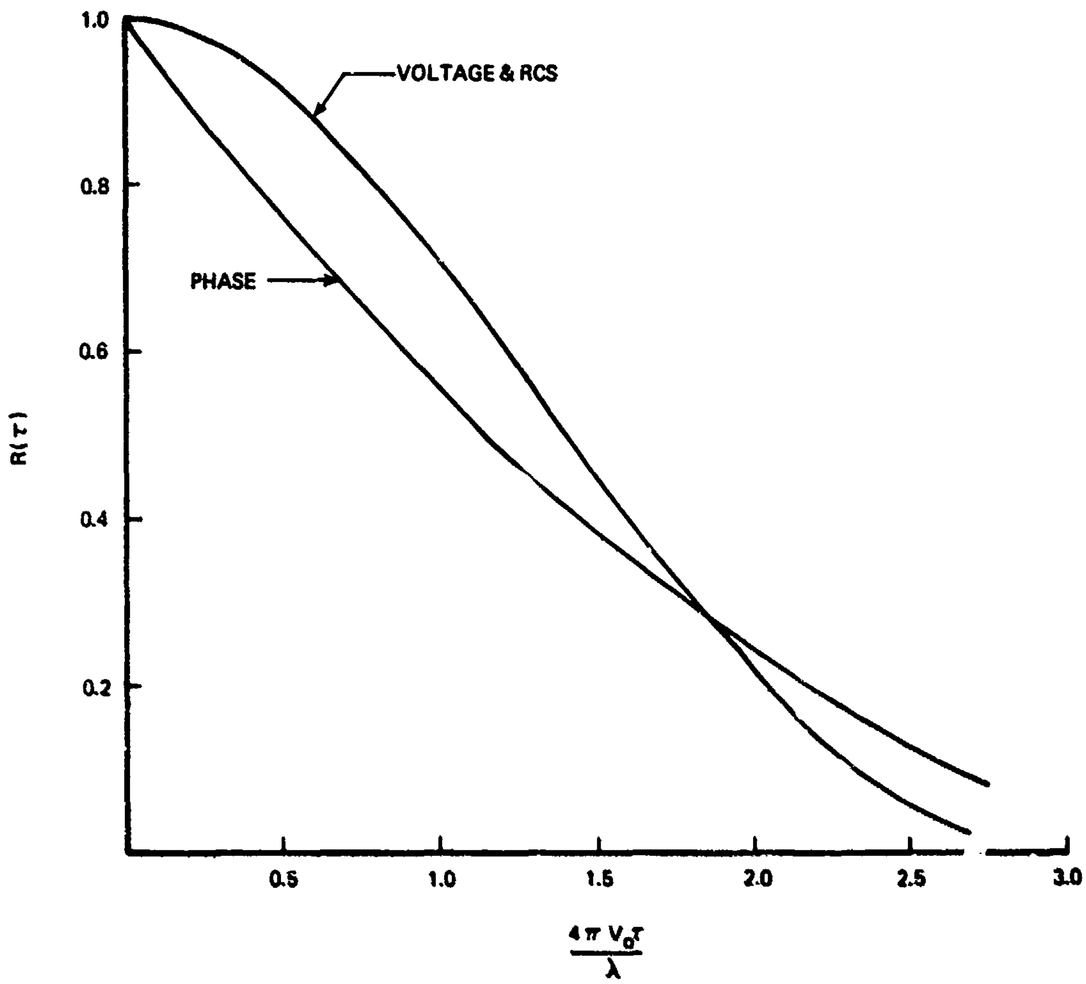

Figure 6. Voltage, RCS and Phase Auto-covariance Functions for Constar' Speed Distribution

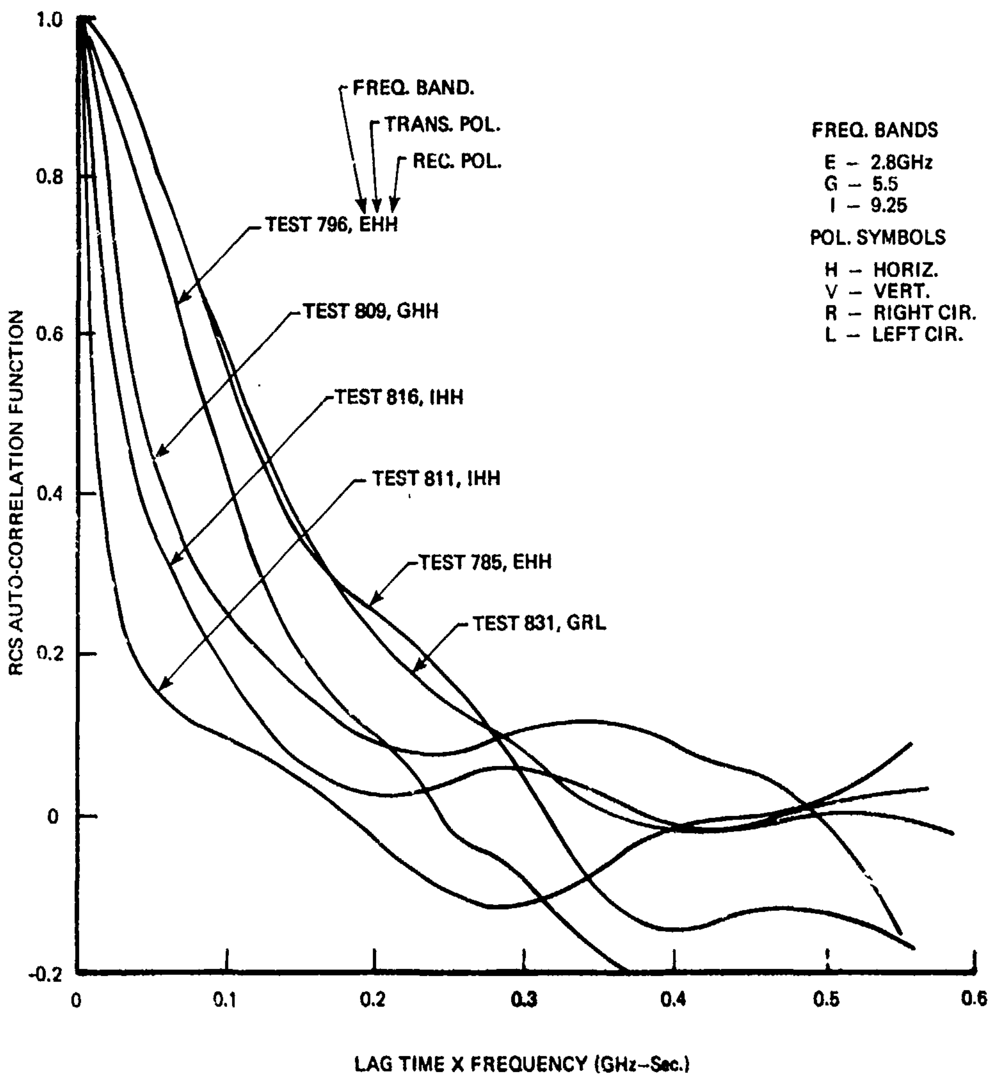

Figure 7. Intensity Auto-correlation Function by Experiment

SECTION V
CONCLUSIONS AND RECOMMENDATIONS

This effort accomplished the task of obtaining mathematical expressions for the first and second order statistics of electromagnetic scattering from chaff clouds. These results can form a basis for studying the effects of chaff clutter fluctuations on different types of advanced radars in a statistical sense. For example, the starting point in the case of noncoherent MTI with a single delay line canceller is the conditional probability for the amplitudes given by

$$p\left(V_2 \middle| V_1;\tau\right) = \frac{p\left(V_1,V_2;\tau\right)}{p\left(V_1\right)}$$
$$= \frac{2V_2}{I_o\left(1-g^2\right)} \; J_o\left[i\,\frac{2gV_1V_2}{I_o\left(1-g^2\right)}\right]$$
$$\exp\left[-\frac{V_2^2+g^2V_1^2}{I_o\left(1-g^2\right)}\right] \tag{5.1}$$

where τ is the interpulse period and V_1 and V_2 are the amplitudes for two successive pulses. The uncancelled output is $|V_2-V_1|$. Herein lies the physical significance of g. If g = 1, there will be perfect cancellation. A quantitative study can now be made in terms of the correlation functions of chaff and the target as parameters. Details of this and similar studies will be presented in subsequent reports.

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